RIGOROUS SOLUTION FOR THE CASE OF ELECTROMAGNETIC WAVE PROPAGATION ALONG A CIRCULAR WAVE GUIDE OF FINITE CONDUCTIVITY

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ABSTRACT. The complete solution for the propagation of electromagnetic waves along a circular wave guide has been worked out. The final solution can be grouped into two parts. One part which can easily be computed in practice has the form of the usual normal mode type solutions, but they are neither orthogonal nor a complete set The other part represented by contour integrals cannot be so easily computed. In the csse of metallic wave guides the contribution by the latter type of fields is indeed negligible, but they do have practical significance and contribute a major part in case of wave guides of small conductivity.

INTRODUCTION

In a previous paper by the author (1951) it was shown that the usual methods of dealing with electromagnetic wave propagation in wave guides, leading to solutions just of the normal mode type are not adequate to describe the complete field of a given source. The additional solutions, though necessary to form a complete solution of the problem, may not be significant in case of metallic wave guides, but they do form a major part in the case of guide walls of finite conductivity, as for example, in the practical case of dielectric wave guides.

The clue to obtain these additional solutions or rather the complete solution of the problem is found in a paper by Sommerfeld (1912), which gives the connection between the normal mode solutions and the residues of a contour integral while discussing the eigenfunction problem for finite regions. And so a solution in the form of a contour integral of the Green's function may give the desired result. In this process, those normal mode solutions, which are permissible for the infinite region under consideration, will automatically appear as the residues of the contour integral.

THEORY

As a general case of the source field, consider the field of an electric dipole of moment P_0 located at the point $Z=o$, $r=r_0$ and $\theta=\theta_0$, within the cylinder of radius a. Let the axis of the dipole make an angle α with the Z axis and let its projection on the $Z=o$ plane make an angle β with the X-axis $(\theta=o)$.

Following Stratton (1941), the expressions for the electric and magnetic field intensities of the dipole in an infinite, homogeneous, isotropic medium are :

$$\bar{E}=\nabla\times\nabla\times(\bar{P}\, e^{ikR/4\pi\epsilon R})$$

$$\bar{H}=\frac{k^{2}}{i\mu\omega}\nabla\times(\bar{P}\, e^{ikR/4\pi\epsilon R}) \tag{1}$$

where ϵ is the dielectric constant, μ is the permeability and k is the propagation constant, ω is the angular frequency of the radiation and $R=\sqrt{r_1^2+Z^2}$ where r_1 is the radial distance from the dipole to the point of observation. The vector P is given by

$$\bar{P}=P_o\,(\bar{i}_r \sin\alpha\cos\theta_1-\bar{i}_\theta \sin\alpha\sin\theta_1+\bar{i}_z\cos\alpha)$$

where θ_1 is the angle between r_1 and the projection of $\bar{P}$ on the $Z=0$ plane. The corresponding field potentials of the source are :

$$\phi_s^{(1)}=\frac{P_0}{4\pi\epsilon}\left(\cos\alpha-\frac{Z}{r_1}\sin\alpha\cos\theta_1\right)e^{ikR/R} \tag{2}$$

and

$$\phi_s^{(2)}=-\frac{P_0}{4\pi\epsilon}\frac{k}{\mu\omega}\frac{\sin\alpha\sin\theta_1}{r_1}e^{ikR} \quad \ldots \tag{3}$$

where the superscripts 1 and 2 refer to the transverse magnetic and transverse electric cases respectively.

The spherical forms in equations (2) and (3) should be converted into cylindrical ones, so as to match the fields at the cylindrical surface ($r=a$) with the result :

$$\phi_s^{(1)}=\frac{P_0}{8\pi\epsilon}\int_{i\infty}^{\infty}\left\{i\cos\alpha\, H_0^{(1)}(ur)+\sin\alpha\cos\theta_1\frac{h}{u}H_1^{(1)}(u_1)\right\}e^{ihZ}\,dh \quad \ldots \tag{4}$$

and

$$\phi_1^{2}=\frac{P_0}{8\pi\epsilon}\frac{k_1^2}{\mu\omega}\sin\alpha\sin\theta_1\int_{i\infty}^{\infty}H^{(1)}(ur)\,e^{ihZ}\frac{dh}{u} \quad \ldots \tag{5}$$

In these integrals the $H^{(1)}$ s are Hankel functions of the first kind and $u=\sqrt{k_1^2-h^2}$ with the path of integration below the branch point at $h=k_1$ in the first quadrant, h is the familiar propagation factor.

The total field potential which is a combination of $\phi_s^{(1)}$ and $\phi_s^{(2)}$ plus the non-singular fields may be written as :

$$\phi_1^{(m)}=\sum^{\infty} e^{in\theta}\int\left\{F_n^{(m)}(h)J_n(ur)-F_n^{(m)}(h)G_n^{(m)}(h)J_n(ur)\right\}e^{ihz}\times dh$$

for o : r_0

$$\phi_n^{(m)\prime}=\sum_{n=-\infty}^{\infty} e^{in\theta}\int F_n^{(m)}(h)\{H_n^{(1)}(ur)-G_n^{(m)}(h)J_n(ur)\}e^{ihz}\times dh$$

for r_0 : $r\leq a$

and

$$\phi_2^{(m)} = \sum_{n=-\infty}^{\infty} e^{in\theta} \int_{l_1} F_n^{(m)}(h)\{H_n^{(1)}(ua) - G_n^{(m)}(h) J_n(ua)\} \times \frac{H_n^{(1)}(vr)e^{ihz}}{H_n^{(1)}(va)} \frac{u^2}{v^2}\, dh$$

for $r \geq a$ ... (6)

In these expressions, the superscript (m) in general can be either 1 or 2 ; $u = \sqrt{k_1^2 - h^2}$ and $v = \sqrt{k_2^2 - h^2}$. Z is assumed to be positive and the path of integration passes below the branch points at $h = k_1$ and $h = k_2$ in figure 1 both u and v having imaginary parts along the path. Further,

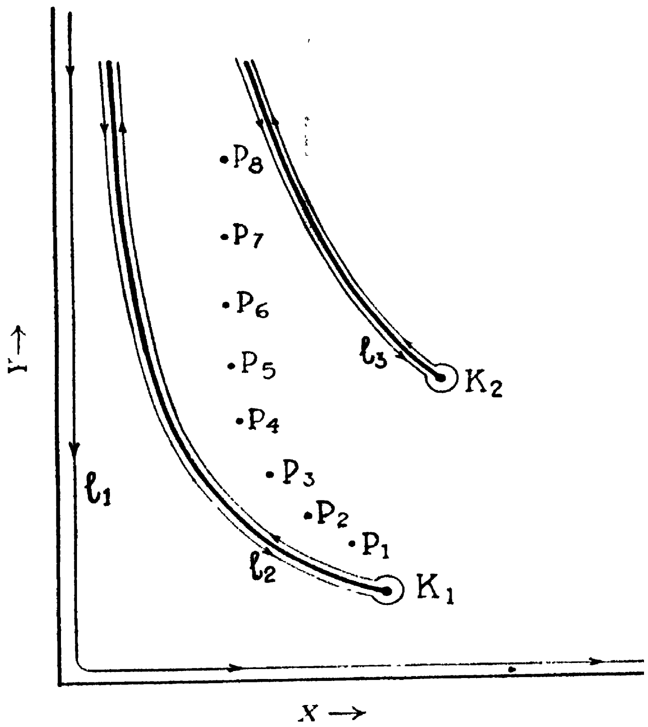

FIG. 1

Path of integration in the $h = X + iY$ plane

$$F_n^{(1)}(h) = \frac{iP_0}{8\pi\epsilon_1} e^{-in\theta_0} \left[\left\{ \cos\alpha - \left(\frac{hn}{u^2 r_0}\right) \sin\alpha \sin\beta \right\} J_n(ur) - \left(\frac{ih}{u}\right) \sin\alpha \cos\beta\, J'_n(ur_0) \right]$$

and

$$F_n^{(2)}(h) = \frac{iP_0}{8\pi\epsilon_1} \frac{k^2}{i\mu_1\omega} e^{-in\theta_0} \left[-\frac{in}{u^2 r_0} \sin\alpha \cos\beta\, J_n(ur_0) - \sin\alpha \sin\beta\, J'_n(ur_0)/u \right.$$

... (7)*

*$\overline{F}_n^{(m)}(h) = F_n^{(m)}(h)$ with the Bessel functions J_n and J_n' replaced by H_n and H'_n respectively. The prime denotes differentiation with respect to the argument.

And further, the co-efficients $G_n^{(m)}(h)$ in the set of equations (6) are given by :

$$G_n^{(1)}(h) = \frac{H_n^{(1)}(ua)}{J_n(ua)} \left\{ 1 + \frac{\mathbf{J}_n(ua) - \mathbf{H}^{(1)}(ua)}{A_n(h)} \times \left[i\mu_1\omega nha^{-2}(u^{-2} - v^{-2})\frac{F_n^{(2)}(h)}{F_n^{(1)}(h)} - f_1(\mu_1\mathbf{J}_n(ua) - \mu_2\mathbf{H}_n^{(1)}(va)) \right] \right\}$$

and

$$G_n^{(2)}(h) = \frac{H_n^{(1)}(ua)}{J_n(ua)} \left\{ 1 + \frac{[\mathbf{J}_n(na) - \mathbf{H}_m^{(1)}(ua)]}{A_n(h)} \left[\frac{k_1{}^2nh}{i\omega a^2\mu_1} (u^{-2} - v^{-2})\frac{F_m^{(2)}(h)}{F_n^{(1)}(h)} - \mu_1\left(\frac{k^2}{\mu_1} \mathbf{J}_n(ua) - \frac{k_2{}^2}{\mu_2}\mathbf{H}_n^{(1)}(va) \right)\right] \right\} \quad \ldots \quad (8)$$

with

$$A_n(h) = \left\{ \frac{k_1{}^2}{\mu_1} \mathbf{J}_n(ua) - \frac{k_2{}^2}{\mu_2} \mathbf{H}_n^{(1)}(ua) \right\} \{\mu_1\mathbf{J}_n(ua) - \mu_2\mathbf{H}_n^{(1)}(va)\} - n^2h^2a^{-4}(u^{-2} - v^{-2})^2 \quad \ldots \quad (9)$$

The functions $\mathbf{J}_n(x)$ and $\mathbf{H}_n^{(1)}(x)$ in equations (8) and (9) are defined by :

$$\mathbf{J}_n^{(x)} \equiv \frac{J_n^1(x)}{xJ_n(x)} \text{ and } \mathbf{H}_n^{(1)}(x) \equiv \frac{H_n^{(1)\prime}(x)}{xH_n^{(1)}(x)} \quad \ldots \quad (10)$$

Equations (8), (9) and (10) result from the continuity of the tangential components E_θ and H_θ at $r=a$, while the continuity of E_z and H_z at $r=a$ is given by the last expression in equation (6). The fields derived from the first two expressions in equation (6) match at $r=r_0$ while the first terms within the brackets in these represent the source field. For $r > a$, the fields represent outgoing waves only.

The integrands in the integrals of equation (6) have branch points at $h=k_1$ and $h=k_2$ and poles P_j at the zeros of $A_n(h)=0$. Figure 1 shows the location of these poles. In drawing this figure the Riemann sheet has been so chosen that the real parts of u and v are positive below and negative above the hyperbolae passing through k_1 and k_2 respectively. The imaginary parts of u and v, however, are positive everywhere. Along an arc at $|h| \longrightarrow \infty$ in the first quadrant the integrands vanish exponentially, and the path of integration will be deformed into the paths l_2 and l_3 around the branch points at k_1 and k_2 plus residues at the poles p_j. As the integrands of the set of equations in (6) are even functions of u the integrals along the contour l_2 will vanish. But as they are not even functions of v the integral along l_3 will give a finite contribution to the total field.

After evaluating the residues at the poles p_j the rigorous solution for the complete field of an electric dipole located inside an infinite circular of radius a may be written in the abbreviated from :

$$\phi_{1,2}^{(m)} = \sum_j B_{1,2}^{(m)}(h_j) + C_{1,2}^{(m)} \quad \ldots \quad (11)$$

where the summation over j includes the sum over all n and for each n the sum over all the roots of $A_n(h_j)=0$ for which the imaginary part of V_j is positive.

Further,

$$B_1^{(m)}(h_j) = D^{(m)}(h_j) e^{in\theta} J_n(u_j r) e^{ih_j Z}$$

$$B_2^{(m)}(h_j) = D^{(m)}(h_j) \frac{u_j^2}{v_j^2} \frac{J_n(u_j a)}{H_n^{(1)}(v_j a)} e^{in\theta} H_n^{(1)}(v_j r) e^{ih_j Z}$$

where,

$$D^{(m)}(h_j) = \frac{-4F_n^{(m)}(h_j) E^{(m)}(h_j)}{\left\{ u_j a J_n(u_j a) \right\}^2 \left[\frac{d}{dh} A_n(h) \right]_{h=h_j}}$$

$$E^{(1)}(h_j) = \left| i\mu\omega \frac{nh_j F_n^{(2)}(h_j)}{a^2 F_n^{(1)}(h_j)} (u^{-2} - v^{-2}) - \frac{k_1^2}{\mu_1} \{\mu_1 \mathbf{J}_n(ua) - \mu_2 \mathbf{H}_n^{(1)}(va) \right.$$

and,

$$E^{(2)}(h_j) = \left[\frac{k_1^2 n h_j F_n^{(1)}(h_j)}{i\mu_1 \omega a^2 F_n^{(2)}(h_j)} (u^{-2} - v^{-2}) \right.$$

$$\left. \cdot \mu_1 \left\{ \frac{k_1^2}{\mu_1} \mathbf{J}_n(ua) - \frac{k_2^2}{\mu_1} \mathbf{H}_n^{(1)}(Va) \right\} \right. \qquad (12)$$

The terms $C_1^{(m)}$ and $C_2^{(m)}$ in equation (11) are given by the first and third expressions respectively in equation (6) with the path of integration l_1 replaced by l_3.

The first term in equation (11) can be easily computed by evaluating the roots h_j, but not so, the second term represented by the C's. For a metallic wave guide with σ_2 large, the second term in equation (11) falls off very rapidly—as fast as $e^{-Z\sqrt{\frac{\sigma_2 \mu_2 \omega}{2}}}$ —for large Z, as a first approximation. But if σ_2 is small as is the case in a dielectric wave guide, the contribution by the Cs can be significant. As an example, if the conductivity σ_1 of the region inside the guide is zero, and that of the guide is small the Cs in equation (11) for the case of a dipole oriented along the axis of the cylinder are given by the integral expressions :

$$C_1^{(1)} = \frac{-P_0 k_1^2 \mu_2}{4\pi\epsilon_1 k_2^2 \mu_1} \int_{k_2}^{i\infty} \frac{4iJ_0(ur) e^{ihZ}\, dh}{\pi \{uaJ_0(ua)\}^2 H_1^{(1)}(va) H_1^{(2)}(va) X^{(1)} X^{(2)}}$$

$$C_2^{(1)} = \frac{-P_0 k_1^2 \mu_2}{4\pi\epsilon_1 k_2^2 \mu_1} \int_{k_2}^{i\infty} \frac{e^{ihZ}}{vaJ_0(ua)} \left\{ \frac{H_0^{(1)}(vr)}{X^{(1)} H_1^{(1)}(va)} - \frac{H_0^{(1)}(vr)}{X^{(2)} H_1^{(2)}(va)} \right\} dh$$

and

$$C_1^{(2)} = C_2^{(2)} = 0 \qquad \ldots \quad (13)$$

where

$$X^{(m)}_{m=1,2} = 1 - \frac{k_1^2 \mu_2 v J_1(ua) H_0^{(m)}(vu)}{k_2^2 \mu_1 u J_0(ua) H_1^{(m)}(va)}$$

and the $H^{(2)}$, s refer to Hankel functions of the second kind.

For Z large, the expressions in equation (13) will simplify to the approximate forms :

$$C_1^{(1)} \approx \frac{P_0 k_1 \mu_2}{4\pi\epsilon_1 k_2^2 \mu_1} \frac{J_0(r\sqrt{k_1^2-k_2^2})}{J_0^2(a\sqrt{k_1^2-k_2^2})} \frac{2k_2 e^{ik_2 Z}}{iZ^2(k_1^2-k_2^2)}$$

and,

$$C_2^{(1)} \approx \frac{P_0}{4\pi\epsilon_1} \frac{k_1^2 \mu_2}{k_2^2 \mu_1} \frac{1}{J_0(a\sqrt{k_1^2-k_2^2})} \frac{e^{(ik_2\sqrt{Z^2+r^2})}}{\sqrt{Z^2+r^2}} \quad \ldots \quad (14)$$

which have the form of dipole type fields reduced in amplitude at the discontinuity $(r=a)$.

Thus the complete field of an electric dipole located inside an infinite circular cylinder can be divided into two parts. The first part represented by the fields $B_{1,2}^{(m)}$ in equation (11) has the form of the usual normal modes, though they are not orthogonal and do not form a complete set. These fields could be easily computed in any practical case under consideration. The second part represented by the fields $C^{(m)}_{1,2}$ in equation (11), however, does not lend itself to easy computation. These latter type of fields are not significant and contribute but a negligible correction term in case of metallic wave guides, but they, however, become important and contribute a major part in case of dielectric wave guides. In the special case of a region of zero conductivity bounded by a guide of small conductivity discussed above, the fields $C^{(m)}_{1,2}$ represent space waves. The field $C_1^{(1)}$, in particular, represents energy from the out-going space wave $C_2^{(1)}$ which has re-entered the region inside the cylinder, because the propagation constant (k_2) of the outer medium $(r > a)$ occurs in its exponential instead of the propagation constant (k_1) of the inner medium, $(r < a)$.

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